Laplace Transform for PDEs

we know $\mathcal{L}\{y(t)\} = \int_0^\infty y(t)e^{-st}\,dt = Y(s)$

$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$

$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$

t is integrated away
replaced w/ s

now let's do the same for $u(x,t)$

$\mathcal{L}\{u(x,t)\} = \int_0^\infty u(x,t)e^{-st}\,dt = U(x,s)$ $\qquad$ x is "constant"

$\mathcal{L}\{u_t(x,t)\} = sU(x,s) - u(x,0)$

$\mathcal{L}\{u_{tt}(x,t)\} = s^2U(x,s) - su(x,0) - u_t(x,0)$

how about $u_x(x,t)$?

$\mathcal{L}\{u_x(x,t)\} = \int_0^\infty u_x(x,t)e^{-st}\,dt$ $\qquad$ x does not get involved here

$= \int_0^\infty \frac{\partial}{\partial x} u(x,t)e^{-st}\,dt$

$$= \frac{\partial}{\partial x}\left[\int_0^\infty u(x,t)\,e^{-st}\,dt\right]$$

$$= \frac{\partial}{\partial x} U(x,s)$$

$$\mathcal{L}\{u_{xx}(x,t)\} = \frac{\partial^2}{\partial x^2} U(x,s)$$

let's do a simple heat eg.

<u>example</u>

$$u_t = u_{xx} \qquad 0 < x < \pi \;,\; t > 0$$

$$u(0,t) = 0$$

$$u(\pi,t) = 0$$

$$u(x,0) = \sin x$$

$$\mathcal{L}\{u_t\} = \mathcal{L}\{u_{xx}\}$$

$$sU(x,s) - u(x,0) = \frac{\partial^2}{\partial x^2} U(x,s) \qquad u(x,0) = \sin x$$

$$sU - \sin(x) = U'' \qquad \text{(prime} \rightarrow \text{deriv. w/ } x\text{)}$$

$U'' - sU = -\sin(x)$ think of it as $y'' - ay = -\sin(x)$

find complementary solution: $U'' - sU = 0$ s is "constant"

$U = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x}$

to find c_1, c_2, use the BCs

$u(0,t) = 0 \rightarrow \mathcal{L}\{u(0,t)\} = \mathcal{L}\{0\} \rightarrow U(x=0) = 0$

$u(\pi,t) = 0 \rightarrow U(x=\pi) = 0$

w/ those we get $0 = c_1 + c_2$

$0 = c_1 e^{\sqrt{s}\pi} + c_2 e^{-\sqrt{s}\pi}$

$\vdots$

$c_1 = c_2 = 0$

now the particular solution

using undetermined coeff: $U_p = A\cos x + B\sin x$

plug into $U'' - sU = -\sin(x)$

⋮

$A = 0, \quad B = \frac{1}{s+1}$

so, $\boxed{U(x,s) = \frac{1}{s+1} \sin x}$

Solution is s-domain

we often want the solution in t-domain

$u(x,t) = \mathcal{L}^{-1}\left\{ \frac{1}{s+1} \sin x \right\}$

s is variable

x is "constant"

$= \sin x \, \mathcal{L}^{-1}\left\{ \frac{1}{s+1} \right\}$

$$\boxed{u(x,t) = e^{-t} \sin x}$$

Laplace transform is particularly effective w/ infinite domains and time-vary boundary conditions

→ separation of variables can't handle these easily example $\quad u_t = u_{xx} \qquad 0 < x < \infty$

$u(0,t) = \sin(t)$ left end temp = $\sin(t)$

$u(x,0) = 0$ initially frozen

$$\mathcal{L}\{u_t\} = \mathcal{L}\{u_{xx}\}$$

$$sU - u(x,0) = U'' \longrightarrow U'' - sU = 0$$

$$U = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x}$$

"hidden" BC at ∞ : temp must be bounded 'at ∞'

$\longrightarrow \quad c_1 = 0$

$$U = c_2 e^{-\sqrt{s}\,x}$$

BC at $x=0$: $\quad u(0,t) = \sin(x)$

$$U(x=0) = \frac{1}{s^2+1}$$

so, $\quad c_2 = \dfrac{1}{s^2+1}$

$$\boxed{U(x,s) = \frac{1}{s^2+1} e^{-\sqrt{s}\,x}}$$ Solution in s-domain

back to t: convolution

$$\mathcal{L}\left\{ \int_0^t f(t-\tau)\, g(\tau)\, d\tau \right\} = FG$$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s^2+1} \right\} = f(t) = \sin(t)$$

$$\mathcal{L}^{-1}\left\{ e^{-\sqrt{s}\,x} \right\} = \frac{x}{2\sqrt{\pi t^3}} e^{-x^2/4t} = g(t)$$

$$\boxed{u(x,t) = \int_0^t \sin(t-\tau) \frac{x}{2\sqrt{\pi \tau^3}} e^{-x^2/4\tau}\, d\tau}$$

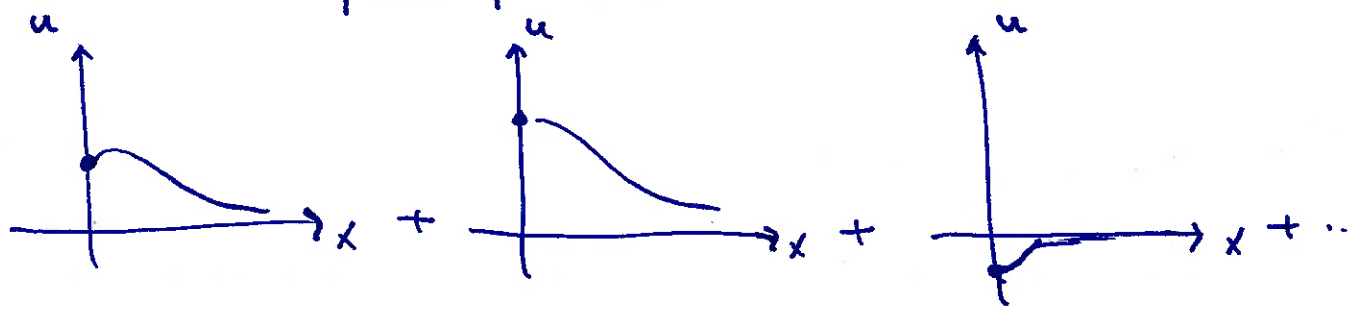
$\sin(t)$: the heat we put in

$\frac{x}{2\sqrt{\pi t^3}} e^{-x^2/4\tau}$: how heat moves in this rod

(heat kernel)

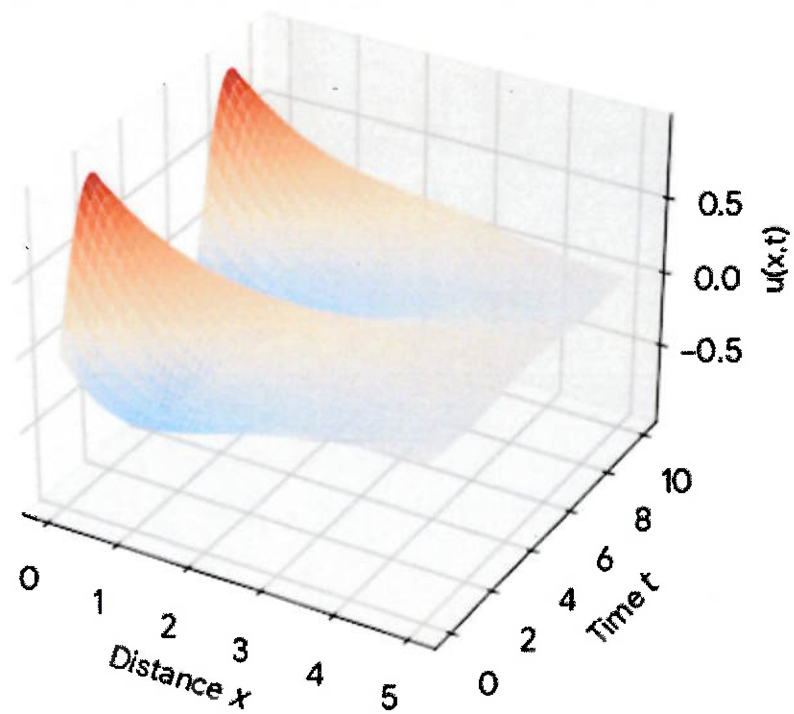
heat kernel is also the impulse response in reaction

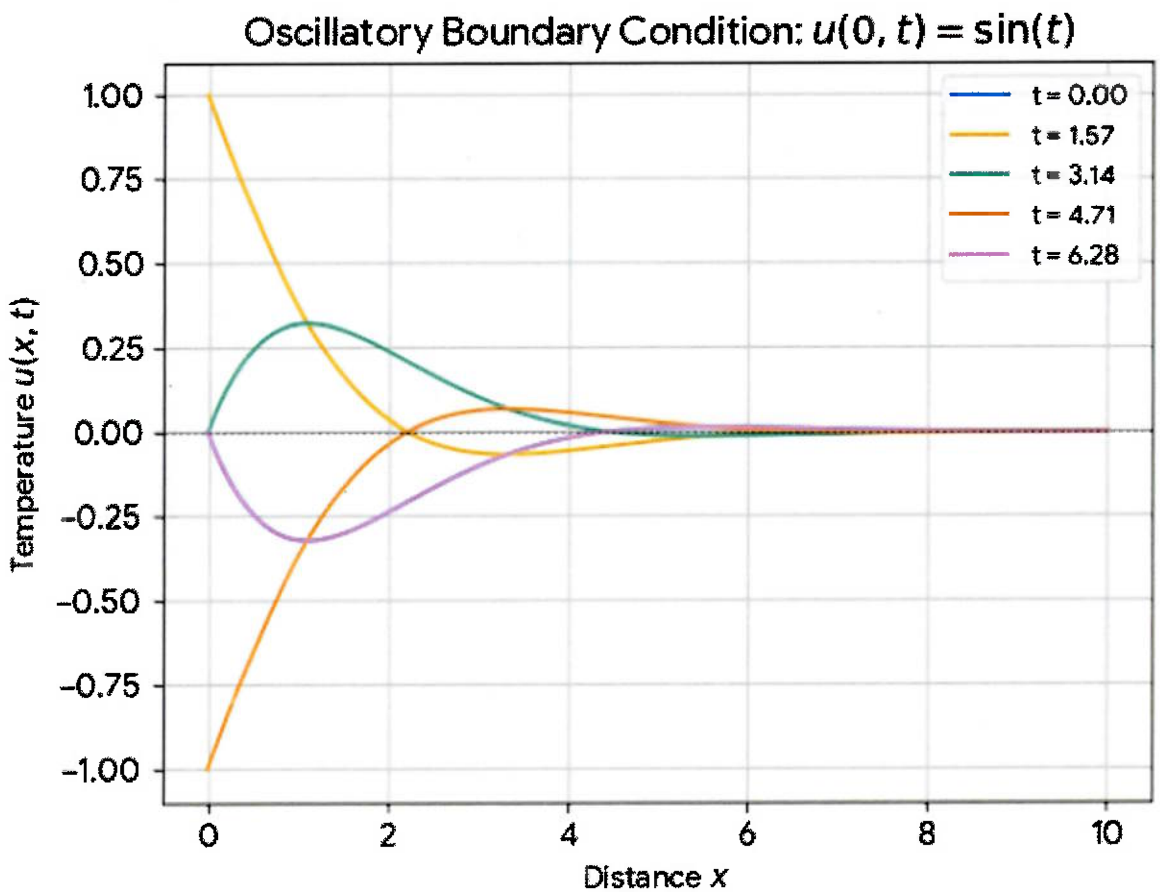
to an impulse of heat



what is measured is the sum (integral) of all those

Thermal Wave Propagation into the Rod





Oscillatory Boundary Condition: $u(0, t) = \sin(t)$
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
2
4
6
8
10
Temperature $u(x, t)$
Distance x
t = 0.00
t = 1.57
t = 3.14
t = 4.71
t = 6.28